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Random-matrix theory and eigenvector statistics of dissipative dynamical systems

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Abstract. We investigate the predictions of random-matrix theory for eigenvector statistics and compare them with numerical results obtained for the dissipative periodically kicked top. Different types of statistics are found. In contrast to conservative dynamics they are connected with the types of eigenvalue of the propagator rather than with the symmetries of Hamiltonian evolution.

Random-matrix theory (RMT) [1, 2] has proven itself very successful in its predictions for eigenvalue statistics of quantum systems with a chaotic classical limit [3–5]. It has been demonstrated that spectra of Hamilton or evolution (Floquet) operators differ in cases of classically regular and chaotic motion. The level-spacing distributions obtained for chaotic systems fit the results obtained from RMT for Gaussian or circular ensembles, respectively. The symmetry properties of the quantum system determine which of the three universality classes—orthogonal, unitary or symplectic—should be used. The fourth universality class, a Poissonian one, corresponds to regular motion. Similar results were obtained for the statistics of eigenvector components [5–13]. It has been shown that the statistical properties of eigenvectors can also be treated as a signature of quantum chaos and the statistics of eigenvectors of chaotic systems conform to the known theoretical distributions for orthogonal, unitary or symplectic ensembles. The main profit from eigenvector analysis is that the data available for statistics are much greater than the number of eigenvalues.

Recently there has been an interest in the application of RMT to quantum dissipative systems. It has been shown that the level spacing distribution of the quantum propagator for the periodically kicked dissipative top coincides with predictions obtained from RMT for an ensemble of arbitrary complex matrices (Ginibre ensemble). The cubic level repulsion found in the case of fully developed chaos (and reasonable high damping) may serve as a signature of quantum chaos since, for regular motion, linear repulsion was observed. Besides, the universality of cubic level repulsion, which is a characteristic of quantum chaos, was observed, that is level repulsion is independent of the symmetry of the Hamiltonian evolution [14, 15].

In this paper we discuss the eigenvector statistics for dissipative systems and compare them with predictions obtained from RMT. As a model of a dissipative physical system we adopt the well known periodically kicked spin system. A detailed description of this model may be found, for example, in [5]. Due to the dissipation the system has to be described by a density matrix ρ. It is assumed that the evolution consists of two separate steps described by operators L and Λ:

$$\rho(t+1) = D\rho(t) = \exp(\Lambda)\exp(L)\rho(t) \tag{1}$$

where $\exp(L)$ represents standard unitary motion

$$\exp(L)\rho = F\rho F^{\dagger}$$

with F defined by

$$F = \exp(-\mathrm{i}\beta J_z)\exp\left(-\mathrm{i}\frac{K_1 J_y^2}{2j}\right)\exp\left(-\mathrm{i}\frac{K_2 J_x^2}{2j}\right)$$

while the generator Λ, describing damping, is obtained from the formula

$$\Lambda\rho = \frac{1}{2j}\{\gamma_1([J^+, \rho J^-] + \mathrm{HC}) + \gamma_2([J^-, \rho J^+] + \mathrm{HC})\}.$$

The dissipative part, which is more general than in [14, 15] (it takes into account the temperature of reservoirs), is chosen so that the squared angular momentum $J^2 = j(j+1)$ is conserved. Moreover, the form of unitary operator F pertains to two classes of Hamiltonian evolution: unitary and orthogonal (we do not discuss here the third class, the symplectic one). Therefore we can investigate whether the class of Hamiltonian evolution influences the eigenvector statistics for a dissipative system. Let us note that, for given value of j, ρ is a $(2j+1)\times(2j+1)$ matrix, while D is a $(2j+1)^2 \times (2j+1)^2$ matrix. Thus, for given j, the computer requirements (storage and time) for a dissipative problem are much greater than for a conservative one.

Let us now consider the eigenvalue problem for the quantum propagator D:

$$D\rho_k = \lambda_k \rho_k. \tag{2}$$

It is convenient to treat eigenvectors ρ_k as $(2j+1)\times(2j+1)$ matrices. Then the eigenvalue problem (2) possesses the following properties:

(i) the eigenvalues of D are either real or come in complex conjugate pairs;

(ii) if λ_k is real, the eigenvector ρ_k is a Hermitian matrix; and

(iii) if λ_k is complex, then $\rho_k^{\dagger}$ is an eigenvector to the eigenvalue λ_k^*.

These statements can be easily deduced from conservation of Hermiticity: $(D\rho) = D\rho^{\dagger}$, the property any generator of dissipation has to fulfil [5].

We now assume that D is a random matrix with the properties just described. In the case of eigenvalue statistics the Ginibre ensemble of random matrices was chosen, but then only complex eigenvalues were considered [15]. Since we are also going to treat eigenvectors corresponding to real eigenvalues another ensemble has to be proposed. One possible choice is the ensemble of random real asymmetric matrices since any generator of dissipation, due to the conservation of Hermiticity, may be represented as a real matrix [5]. Statistical properties of eigenvalues of such random matrices were recently discussed by Lehmann and Sommers [16]. Another choice is motivated by the construction of generator D (1). We consider an ensemble of complex matrices of the following form: $R\cdot(S\otimes S^*)$, where R is a real asymmetric $n^2 \times n^2$ matrix and S is a member of circular ensemble of $n \times n$ matrices. The product of two $n \times n$ matrices is a $n^2 \times n^2$ matrix defined by the formula $(A\otimes B)_{(i-1)*n+k,(j-1)*n+l} = A_{i,j}B_{k,l}$.

The only characteristic of an eigenvector of D is the norm itself. Consider the real eigenvalue λ_k, then there are $N = 2j+1$ real elements x_n on the diagonal of the matrix ρ_k and $M = j(2j+1)$ complex off-diagonal components $c_m = y_m + \mathrm{i}z_m$ such that

$$\sum_{n=1}^{N} x_n^2 + 2\sum_{m=1}^{M} |c_m|^2 = 1.$$

The joint probability density for the components of a Hermitian eigenvector must therefore be

$$P_r(\{x_n\}, \{c_m\}) = \text{constant} \times \delta\left(1 - \sum_{n=1}^{N} x_n^2 - 2\sum_{m=1}^{M} |c_m|^2\right) \tag{3}$$

the constant being fixed by normalization. The quantities convenient to compare with numerical data are the reduced densities of x_1

$$P_{rr}(A) = \int \mathrm{d}x_1 \ldots \mathrm{d}x_N \, \mathrm{d}^2c_1 \ldots \mathrm{d}^2c_M \, \delta(A - x_1^2) P_r(\{x_n\}, \{c_m\}) \tag{4}$$

and of c_1

$$P_{rc}(A) = \int \mathrm{d}x_1 \ldots \mathrm{d}x_N \, \mathrm{d}^2c_1 \ldots \mathrm{d}^2c_M \, \delta(A - |c_1|^2) P_r(\{x_n\}, \{c_m\}). \tag{5}$$

For a complex eigenvalue λ_k the situation is different as there are $N = (2j+1)^2$ independent complex components of the corresponding eigenvector ρ_k. We notice that the same numbers define the eigenvector $\rho_l = \rho_k^\dagger$ corresponding to the eigenvalue $\lambda_l = \lambda_k^*$. The situation is similar to Kramers' degeneracy in the Hamiltonian case. However, since the eigenvalues are different, the corresponding eigenvectors are uniquely defined (up to phase factor) by any diagonalization routine and the reduced density of c_1 is a well defined quantity:

$$P_c(A) = \int \mathrm{d}^2c_1 \ldots \mathrm{d}^2c_N \, \delta(A - |c_1|^2) P_c(\{c_m\}) \tag{6}$$

with the corresponding distribution $P_c(\{c_m\})$

$$P_c(c_m) = \text{constant} \times \delta\left(1 - \sum_{m=1}^{N} |c_m|^2\right). \tag{7}$$

The integration in (4)–(6) may be performed explicitly with the help of the following formulae [9]. Let

$$P^d(x) = \pi^{-d/2}\Gamma(d/2)\delta\left(1 - \sum_{n=1}^{d} x_n^2\right) \tag{8}$$

be the properly normalized joint distribution. Integrating out $d-l$ of the variables x_n yields the result

$$P^{d,l}(\{x\}) = \pi^{-l/2}\frac{\Gamma(d/2)}{\Gamma((d-l)/2)}\left(1 - \sum_{n=1}^{d-l} x_n^2\right)^{(d-l-2)/2}. \tag{9}$$

A direct application of this formula leads to the following densities:

$$P_{rr}(A) = (\pi A)^{-1/2}\frac{\Gamma(N/2)}{\Gamma((N-1)/2)}(1 - A)^{(N-3)/2} \tag{10}$$

$$P_{rc}(A) = 2\frac{\Gamma(N/2)}{\Gamma(N/2-1)}(1 - 2A)^{(N-4)/2} \tag{11}$$

$$P_c(A) = (N-1)(1-A)^{N-2} \tag{12}$$

where $N = (2j+1)^2$.

For $N \gg 1$ the mean value $\langle A \rangle$ is proportional to $1/N$. It is much more convenient to use the rescaled variable y with unit mean value $\langle y \rangle = 1$ [13]. With the help of some algebra the above distributions take the following forms:

$$P_{rr}(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \tag{13}$$

$$P_{rc}(y) = P_c(y) = e^{-y}. \tag{14}$$

One easily notices that (13) and (14) are the distributions obtained for a circular orthogonal ensemble and a circular unitary ensemble, respectively.

We have calculated eigenvectors of D for different values of the parameters. In all computations the angular momentum number j was of order of 10. This value is relatively small especially with respect to the Hamiltonian case where j is usually of order of 100. But for our dissipative model even such a small value as $j = 10$ leads to a diagonalization of the two matrices 221×221 and 220×220, respectively. We notice that considering the top corresponding to the symplectic universality class [5] would lead to the diagonalization of the matrix 441×441. Besides, $j = 10$ was sufficient to obtain excellent agreement with the predictions of RMT for the eigenvalue statistics [14]. Figure 1 presents the histograms obtained. Figures 1(a) and (b) refer to the diagonal and off-diagonal components of eigenvectors corresponding to real eigenvalues, while figure 1(c) presents the results for eigenvectors corresponding to complex eigenvalues ($j = 9.0$, $\gamma_1 = 0.1$, $\gamma_2 = 0.25$, $\beta = 1.7$, $K_2 = 0.0$ and, in order to smooth the histograms, ten different values of K_1 between 6.0 and 8.7 were taken). As in [13] logarithmic scales were used to emphasize the differences between the two types of statistics. The full curve describes the distribution corresponding to the RMT predictions and the broken curve represents the second of the theoretically predicted statistics. We notice that the number of real eigenvalues is of order $(2j + 1)$, therefore the first two histograms are not as smooth as the third one. Moreover, in drawing histograms for real eigenvalues we dropped the eigenvector corresponding to the unit eigenvalue. It is the asymptotic state of the system and has a specific form. Let us note that the fitting of numerical data to the corresponding theoretical curves is not perfect. We think that this is caused by the fact that the propagator is a matrix which is not completely random. Only F can be considered as random, i.e. $(2j + 1)^4$ elements of e^L are determined by $(2j + 1)^2$ elements of the random matrix F. Moreover, we have checked that the accuracy of fitting does not depend on j, at least for $7.0 \leqslant j \leqslant 12.5$.

The parameters in figure 1 correspond to the orthogonal universality class of the operator F. Figure 2 presents a comparison of the statistics of the orthogonal and unitary universality classes for complex eigenvalues. Both classes differ only by the value of constant K_2 which is equal to 0.0 and 1.0 for the orthogonal and unitary cases, respectively (all the remaining parameters of F are the same as in figure 1). It is intuitively clear that with the damping going to zero the statistics should approach, in some way, those obtained for the conservative evolution. Therefore in figures 2(a)–(c) we compare the histograms pertaining to the two classes for three different values of damping: ($\gamma_1 = 0.1$, $\gamma_2 = 0.25$), ($\gamma_1 = 0.005$, $\gamma_2 = 0.0125$) and ($\gamma_1 = 0.13 \times 10^{-4}$, $\gamma_2 = 0.31 \times 10^{-4}$), respectively. One can see that the differences between the universality classes are suppressed with increased damping, so that both histograms in figure 2(a) are, in principle, indistinguishable. An analogous result was obtained for eigenvalue statistics [15]. On the other hand, for very small damping (figure 2(c)) both histograms differ considerably. A comment concerning the second hump in the histogram in figure 2(c) is necessary. Due to the structure of D approximately half

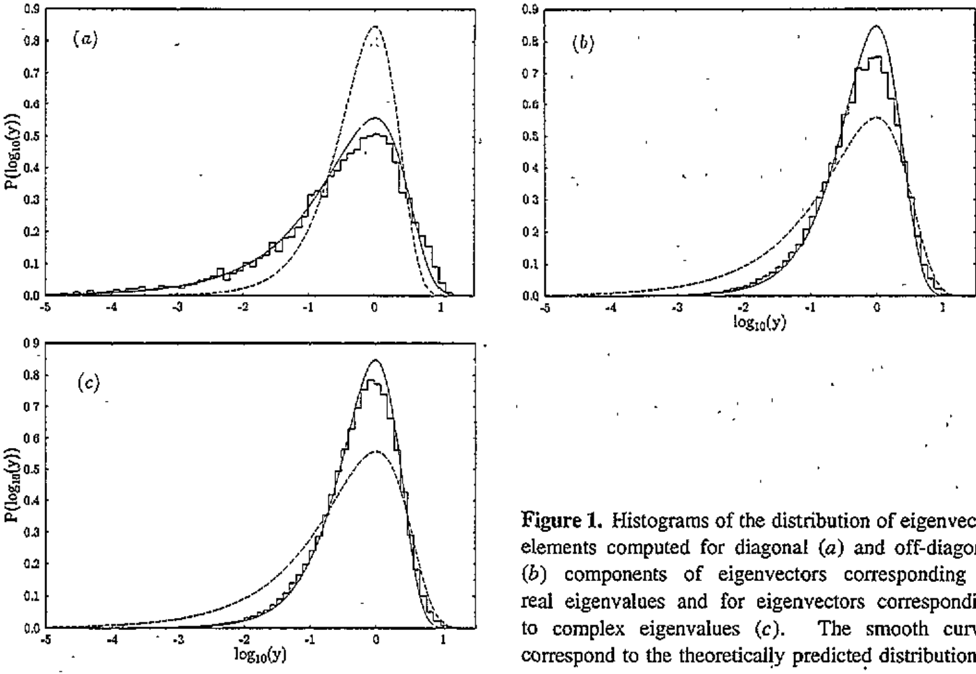

Figure 1. Histograms of the distribution of eigenvector elements computed for diagonal (*a*) and off-diagonal (*b*) components of eigenvectors corresponding to real eigenvalues and for eigenvectors corresponding to complex eigenvalues (*c*). The smooth curves correspond to the theoretically predicted distributions.

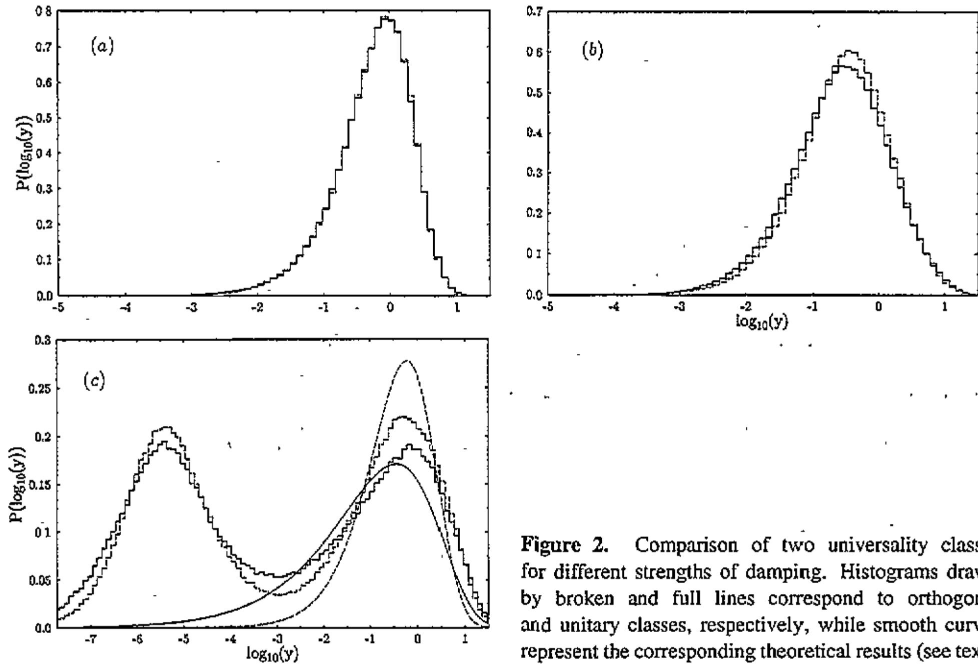

Figure 2. Comparison of two universality classes for different strengths of damping. Histograms drawn by broken and full lines correspond to orthogonal and unitary classes, respectively, while smooth curves represent the corresponding theoretical results (see text).

the components of any eigenvector ρ_k are equal to zero. For zero-damping the eigenvectors of D are constructed from eigenvectors of F in the following way:

$$D|\psi_k\rangle\langle\psi_l| = e^{\beta_k-\beta_l}|\psi_k\rangle\langle\psi_l| \tag{15}$$

where

$$F|\psi_k\rangle = e^{\beta_k}|\psi_k\rangle \tag{16}$$

and the structure of F implies that either j or $j+1$ components of $|\psi_k\rangle$ are zero. Thus for zero-damping only about $(2j+1)^2/4$ components of any eigenvector ρ_k may be non-zero, and additional $(2j+1)^2/4$ components become non-zero with increased damping. These additional components form the left-sided hump in figure 2(c). The other hump should be described (in the limit of zero-damping) by one of the following distributions:

$$P_{\mathrm{COE}}(z) = \frac{1}{\pi\sqrt{z}} K_0(\sqrt{z}) \tag{17}$$

$$P_{\mathrm{CUE}}(z) = 2K_0(2\sqrt{z}) \tag{18}$$

where K_0 is the modified Bessel function. Formulae (17) and (18) follow directly from the rules for the distribution of a product of random variables:

$$P(z) = \int \mathrm{d}x\,\mathrm{d}y\; p(x)p(y)\delta(xy-z)$$

where for $p(y)$ we used either (13) for P_{COE} or (14) for P_{CUE}. Both resulting distributions are presented in figure 2(c) as smooth (broken and full) curves. In fact we compare histograms with rescaled distributions $P(z)/2$.

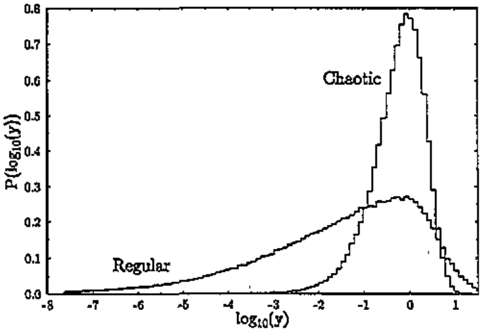

Figure 3. Comparison of eigenvector statistics for chaotic and regular motion. For chaotic motion all data are the same as in figure 1(c) and, for regular motion ten different values of K_1 between 1.0 and 2.35 were taken.

We would like to conclude with the question of signatures of quantum chaos. In figure 3 we show histograms of $P_c(y)$ for parameters corresponding to classical chaos and classically regular motion. As can be seen from this picture there is a considerable difference between these two cases. The regular case can be characterized by a long tail which is also typical for the pure Hamiltonian case [13]. Therefore the eigenvector statistics may serve as a signature of quantum chaos. In some respect it is an even better criterion than eigenvalue statistics. Although the numerical computation of eigenvectors is more time consuming than computations of eigenvalues alone, one obtains more data to compare with theoretical predictions.

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